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Dynamic displacement response of beam-type structures to moving line loads

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Abstract

A closed-form displacement response of beam-type structures to moving line loads is proposed in this paper. Green's function of the beam on an elastic foundation is obtained by means of Fourier transform. The theory of linear partial differential equation is used to represent the displacement of the beam in terms of convolution of the Green's function. To evaluate this convolution analytically, the theory of complex function is employed to seek the poles of the integrand of the generalized integral. All the poles are identified and given in a closed form. Theorem of residue is then utilized to represent the generalized integral using contour integral in the complex plane. Closed-form displacement is provided and numerical computation is performed. The numerical results show that maximum displacement of a beam with material damping occurs behind the moving load. Also, the maximal dynamic displacement reaches its maximum as the load moves at the critical speed.

Keywords: Beam; Moving load; Elastic foundation; Hysteretic damping; Green's function; Fourier transform; Theorem of residue

1. Introduction

The response of beam-type structures to moving loads has been studied extensively over the past several decades (Fryba, 1977). Considerable research has been conducted to investigate the displacement response of beam under moving loads (Kenney, 1954; Achenbach and Sun, 1965; Florence, 1965; Steele, 1967, 1968). A growing interest on this topic also arises in railway and highway industries in recent years because beam-type structures can be used as simplified physical models of rail-track and pavement (Adams and Bogy, 1975; Mulcahy, 1973; Huang, 1977; Choros and Adams, 1979; Saito and Terasawa, 1980; Jezequel, 1981; Patil, 1988; Hardy and Cebon, 1994; Kim and Roesset, 1996; Sun and Deng, 1997).

An interesting aspect of this topic is that when the foundation is modeled as an elastic foundation, a critical velocity is found existing for the moving load, which may cause significant variation of the response of the beam (Kenney, 1954). Waves excited by a moving load with supercritical velocity propagate in a different way as they do when the load velocity is subcritical. The critical velocity is also predicted and

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observed by other researchers either through electrical analogy experiment (Criner and McCann, 1953) or through theoretical analysis (Patil, 1988; Sun, 1998).

Because high-speed vehicles (e.g., trains and automobiles) are getting extensively adopted as the surface transportation carriers, considerable attention has been paid to the response of transportation infrastructure under high-speed loading conditions (Elattary, 1991; Lee, 1994; Pan and Atluri, 1995; Sun, 1998; Sun and Greenberg, 2000). In most studies the foundation of the beam is often assumed as an elastic Winkler foundation. Under this assumption, the critical velocity can be given as $(4KEI/m^2)^{1/4}$ (Criner and McCann, 1953; Kenney, 1954; Sun, 1996; Kim and Roesset, 1996). It has been demonstrated that, moving at the critical speed, the load will excite the beam to reach an infinite displacement (Kenney, 1954; Sun, 1998). In reality, however, damping exists in any physical system and kinetic energy dissipates because of the effect of damping after a period of time. It can be expected that the unrealistic infinite displacement will disappear if damping effect is taken into consideration in the mathematical model of beams subjected to moving loads.

In this paper we considered the effect of material damping proposed by Foinquinos and Roesset (1995) on dynamic displacement response of a flexible beam resting on an elastic foundation subjected to a moving line load. A closed-form representation of the displacement response of the beam is obtained using Fourier transform and the theorem of residue.

This paper is organized as follows. In Section 2, the governing equation of a flexible beam to external load is provided and Green's function of the beam is obtained using Fourier transform. In Section 3, response of the beam to a moving constant load is constructed by convoluting Green's function according to the theory of linear partial differential equation. In Section 4, the roots of the characteristic equation are identified as poles when integrating the generalized integral obtained in Section 3. In Section 5, we obtain the closed-form displacement of the beam by using the theorem of residue to carry out the integral in a complex plane. In Section 6 numerical results are provided to illustrate the shape of the displacement of beam to moving loads. Conclusions are drawn in Section 7 in which summarized results and findings are presented.

2. The Greens function of the beam

The governing equation of a flexible beam on an elastic foundation can be given as (Sun and Deng, 1997; Sun, 1998, 2001)

$$EI\frac{\partial^4 y(x,t)}{\partial x^4} + Ky + m\frac{\partial^2 y(x,t)}{\partial t^2} = F(x,t) \tag{1}$$

where $y(x,t)$ represents the displacement of the beam, x represents the traveling direction of a moving load, and t represents time. Also, EI is rigidity of the beam, E is Young's modulus of elasticity, I is moment of inertia of the beam, m is unit mass of the beam, and $F(x,t)$ is impressed external loads. A moving line load can be expressed by

$$F(x,t) = P\frac{H[r_0^2 - (x-vt)^2]}{2r_0} \tag{2}$$

where r_0 is the half length of the line load, and P is the amplitude of the applied load. Also, $H(\cdot)$ is the Heaviside step function defined by

$$H(x-x_0) = \begin{cases} 0 & \text{for } x < x_0 \\ 1/2 & \text{for } x = x_0 \\ 1 & \text{for } x > x_0 \end{cases} \tag{3}$$

Given an initial condition of the beam, Eqs. (1) and (2) constitute a complete mathematical description of the problem considered here. In this paper, we assume that the beam is at rest initially. In other words, the initial condition of the current problem is zero.

According to the theory of mathematical–physical equation (Morse and Keshbach, 1953), Green's function is the fundamental solution of a partial differential equation. For the current problem, Green's function of the beam corresponds to the solution of Eq. (1) provided that the external load is characterized by Dirac-delta function

$$F_{(\delta)}(x,t) = \delta(x-x_0)\delta(t-t_0) \tag{4}$$

in which $\delta(\cdot)$ is Dirac-delta function defined by

$$\int_{-\infty}^{\infty} \delta(x-x_0) f(x)\,\mathrm{d}x = f(x_0) \tag{5}$$

Define two-dimensional Fourier transform and its inversion as

$$\widetilde{f}(\xi,\omega) = \mathbf{F}[f(x,t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,t)\exp[-\mathrm{i}(\xi x+\omega t)]\,\mathrm{d}x\,\mathrm{d}t \tag{6a}$$

$$f(x,t) = \mathbf{F}^{-1}[\widetilde{f}(\xi,\omega)] = (2\pi)^{-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widetilde{f}(\xi,\omega)\exp[\mathrm{i}(\xi x+\omega t)]\,\mathrm{d}\xi\,\mathrm{d}\omega \tag{6b}$$

where $\mathbf{F}[\cdot]$ and $\mathbf{F}^{-1}[\cdot]$ represent two-dimensional Fourier transform and its inversion, respectively. To solve Green's function, taking Fourier transform on both sides of Eq. (1) gives

$$EI\xi^4\widetilde{G} + K\widetilde{G} - m\omega^2\widetilde{G} = \widetilde{F}_\delta(\xi,\omega) \tag{7}$$

where $\widetilde{F}_\delta(\xi,\omega)$ is the Fourier transform of $F_\delta(x,t)$, and dynamic displacement response $y(x,t)$ has been replaced by the symbol $\widetilde{G} = \widetilde{G}(\xi,\eta;x_0,t_0)$ to indicate the Fourier transform of Green's function. Also, the following property of Fourier transform is used in the derivation of Eq. (7)

$$\mathbf{F}[f^{(n)}(t)] = (\mathrm{i}\omega)^n\mathbf{F}[f(t)] \tag{8}$$

Since $\widetilde{F}_\delta(\xi,\eta)$ is the representation of $F_\delta(x,t)$ in the frequency domain, it is necessary to evaluate the Fourier transform of $F_\delta(x,t)$. This can be implemented by taking Fourier transform on both sides of Eq. (5)

$$\widetilde{F}_\delta(\xi,\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-x_0)\delta(t-t_0)\exp[-\mathrm{i}(\xi x+\omega t)]\,\mathrm{d}x\,\mathrm{d}t = \exp[-\mathrm{i}(\xi x_0+\omega t_0)] \tag{9}$$

in which Dirac-delta function's property, i.e., Eq. (5), is utilized for evaluating the above integral. Substituting this result into Eq. (7) and realizing that Eq. (7) is an algebraic equation, it is straightforward to see

$$\widetilde{G}(\xi,\omega;x_0,t_0) = \exp[-\mathrm{i}(\xi x_0+\omega t_0)](EI\xi^4+K-m\omega^2)^{-1} \tag{10}$$

Green's function given by Eq. (10) is described in the frequency domain and should be converted to the time domain. To this end, take inverse Fourier transform on both sides of Eq. (10)

$$G(x,t;x_0,t_0) = (2\pi)^{-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\exp\{\mathrm{i}[\xi(x-x_0)+\omega(t-t_0)]\}}{EI\xi^4+K-m\omega^2}\,\mathrm{d}\xi\,\mathrm{d}\omega \tag{11}$$

Formula (11) is Green's function of the beam on an elastic foundation. Green's function serves as the fundamental solution of a partial differential equation. It can be very useful as will see later on when dealing with linear systems.

3. Integral representation of the solution

According to the theory of linear partial differential equation (Morse and Keshbach, 1953; Sun and Greenberg, 2000), the solution of Eqs. (1) and (2) can be constructed by integrating the Green's function in all the dimensions. Mathematically,

$$y(x,t) = \int_{-\infty}^{t}\int_{-\infty}^{\infty} F(x_0,t_0)G(x,t;x_0,t_0)\,\mathrm{d}x_0\,\mathrm{d}t_0 \tag{12}$$

Taking Eqs. (2) and (11) into Eq. (12) gives

$$y(x,t) = \int_{-\infty}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{PH[r_0^2-(x_0-vt_0)^2]\exp\{\mathrm{i}[\xi(x-x_0)+\omega(t-t_0)]\}}{(2\pi)^2 2r_0EI(\xi^4+\overline{K}-\overline{m}\omega^2)}\,\mathrm{d}\xi\,\mathrm{d}\omega\,\mathrm{d}x_0\,\mathrm{d}t_0 \tag{13}$$

where $\overline{K} = K/EI$ and $\overline{m} = m/EI$. Since we have

$$\int_{-\infty}^{\infty} \frac{H[r_0^2-(x_0-vt_0)^2]\exp(-\mathrm{i}\xi x_0)}{2r_0}\,\mathrm{d}x_0 = \int_{vt_0-r_0}^{vt_0+r_0} \frac{\exp(-\mathrm{i}\xi x_0)}{2r_0}\,\mathrm{d}x_0 = \frac{\sin r_0\xi}{r_0\xi}\exp(-\mathrm{i}\xi vt_0) \tag{14a}$$

$$\int_{-\infty}^{t} \exp[-\mathrm{i}(\omega+v\xi)t_0]\,\mathrm{d}t_0 = 2\pi\delta(\omega+v\xi) \tag{14b}$$

substituting Eqs. (14a) and (14b) into Eq. (13) and reapplying the property (5) gives

$$y(x,t) = \frac{P}{2\pi EI}\int_{-\infty}^{\infty} \frac{\sin(r_0\xi)\exp[\mathrm{i}\xi(x-vt)]}{r_0\xi(\xi^4-\overline{m}v^2\xi^2+\overline{K})}\,\mathrm{d}\xi \tag{15}$$

So far, we have obtained the integral representation of dynamic displacement response of the beam under a moving line load. For a realistic physical system, damping effect always exists. When foundation damping is considered, according to Sun and Deng (1997), Sun (1996, 1998, 2000) and Kim and Roesset (1996, 1998), a viscous term $\mathrm{i}\overline{C}v\xi$ should be added into the denominator of the integrand of Eq. (15). However, for asphalt concrete pavement, it is also appropriate to consider damping effect in pavement material itself. In this case, linear hysteretic material damping proposed by Foinquinos and Roesset (1995) can be a reasonable model to characterize the material damping effect. For instance, the same material damping of linear nature has been adopted by Kim and Roesset (1996, 1998) to model a plate subjected to a moving load. According to their study (Foinquinos and Roesset, 1995; Kim and Roesset, 1996), a viscous term $-\mathrm{i}2D\overline{K}$ should be added into the denominator of the integrand of Eq. (15) such that

$$y(x,t) = \frac{P}{2\pi EI}\int_{-\infty}^{\infty} \frac{\sin(r_0\xi)\exp[\mathrm{i}\xi(x-vt)]}{r_0\xi(\xi^4-\overline{m}v^2\xi^2+\overline{K}-\mathrm{i}2D\overline{K})}\,\mathrm{d}\xi \tag{16}$$

Expression (16) can be further evaluated using complex function techniques. In the following sections, the theorem of residue is employed to carry out this integration.

4. Roots of the characteristic equation

Before the integration (16) is further evaluated, it is necessary to investigate the roots of the characteristic equation of this type

$$\xi(\xi^4-\overline{m}v^2\xi^2+\overline{K}-\mathrm{i}2D\overline{K}) = 0 \tag{17}$$

Here, we assume that none of the parameters $\overline{m}$, $\overline{K}$, $\overline{C}$, D and v are zero. Characteristic Eq. (17) is a fifth order algebraic equation with parameters of the beam, the foundation and the load. The roots of this equation depend upon the distribution and combination of these parameters. Since an imaginary term existing in this equation, no real-valued roots except $\xi = 0$ exist for Eq. (17). So we only consider the complex roots for equation $\xi^4 - \overline{m}v^2\xi^2 + \overline{K} - \mathrm{i}2D\overline{K} = 0$.

Define the complex root of Eq. (17) as $\xi^2 = z = z_\mathrm{R} + \mathrm{i}z_\mathrm{I}$ ($z_\mathrm{R}, z_\mathrm{I} \in \mathrm{Re}$, and $z_\mathrm{I} \neq 0$). Substituting it into Eq. (17) and comparing the real and imaginary parts give the following quadratic equation systems

$$z_\mathrm{R}^2 - z_\mathrm{I}^2 - \overline{m}v^2 z_\mathrm{R} + \overline{K} = 0 \tag{18a}$$

$$2z_\mathrm{R}z_\mathrm{I} - \overline{m}v^2 z_I - 2D\overline{K} = 0 \tag{18b}$$

From Eq. (18b) we have

$$z_\mathrm{R} = \overline{m}v^2/2 + D\overline{K}/z_\mathrm{I} \tag{19}$$

Replacing z_R in Eq. (18a) by Eq. (19) gives

$$z_\mathrm{I}^4 + (\overline{m}^2v^4/4 - \overline{K})z_\mathrm{I}^2 - D^2\overline{K}^2 = 0 \tag{20}$$

The determinant of Eq. (20) is $\varDelta = (\overline{m}^2v^4/4 - \overline{K})^2 + 4D^2\overline{K}^2 > 0$. Therefore, there exist real roots for Eq. (20) and they can be given by

$$z_\mathrm{I}^2 = [(\overline{K} - \overline{m}^2v^4/4) \pm \varDelta^{1/2}]/2 \tag{21}$$

Since z_I^2 should be positive valued, only the following root is valid and adopted for the imaginary part of z

$$z_\mathrm{I} = \pm\{[(\overline{K} - \overline{m}^2v^4/4) + \varDelta^{1/2}]/2\}^{1/2} \tag{22}$$

Substituting Eq. (22) into Eq. (19) gives the real part of z

$$z_\mathrm{R} = \overline{m}v^2/2 \pm D\overline{K}/\{[(\overline{K} - \overline{m}^2v^4/4) + \varDelta^{1/2}]/2\}^{1/2} \tag{23}$$

Remember $\xi^2 = z = z_\mathrm{R} + \mathrm{i}z_\mathrm{I}$ and we can rewrite ξ^2 as

$$\xi^2 = (z_\mathrm{R}^2 + z_\mathrm{I}^2)^{1/2}\mathrm{e}^{\mathrm{i}\theta} \quad \text{with } \tan\theta = z_\mathrm{I}/z_\mathrm{R} \tag{24}$$

Hence, it is straightforward to give the expression for ξ

$$\xi = (z_\mathrm{R}^2 + z_\mathrm{I}^2)^{1/4}\mathrm{e}^{\mathrm{i}(n\pi+\theta/2)} \quad \text{with } n = 0 \text{ and } 1 \tag{25}$$

5. Closed-form representation of the solution

Now we have obtained all the roots of the characteristic Eq. (17). Fig. 1 plots the location of these poles in the complex ξ-plane. Four complex roots are respectively located in different regions of the complex ξ-plane and a single real root $\xi = 0$ is located exactly at the origin. In general, these four complex poles are distributed in the complex ξ-plane in such a way that a couple of them are located in the upper half-plane and the other couple of them in the lower half-plane. Two cases need to be discussed. One case is the displacement response of the beam in front of the moving load. The other case is the displacement response of the beam behind the moving load.

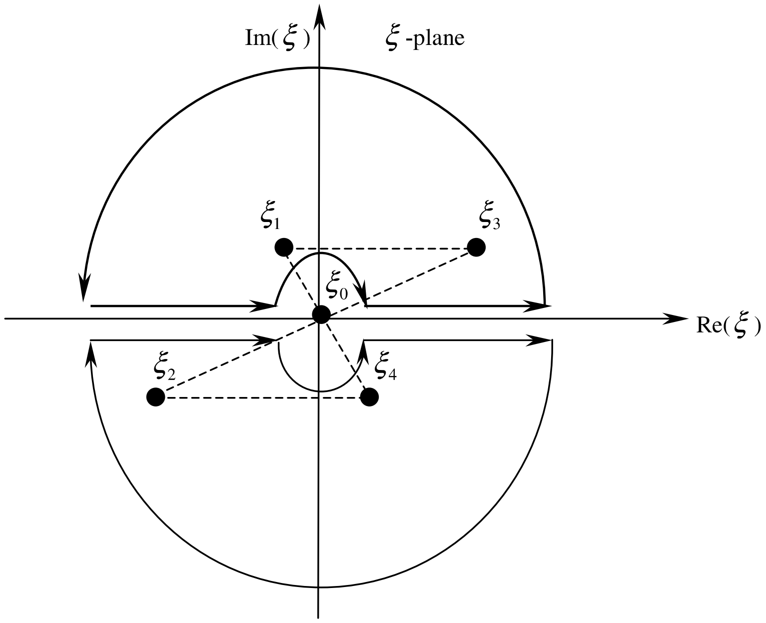

Fig. 1. Contour and poles for evaluating the complex function.

First we consider the solution where $x - vt \geqslant 0$. We are now able to rewrite displacement (17) as

$$y(x,t) = \frac{P}{2\pi EIr_0} \int_{-\infty}^{\infty} \frac{\sin r_0\xi \exp[i\xi(x - vt)]}{\prod_{j=0}^{4}(\xi - \xi_j)} d\xi \tag{26}$$

where ξ_j $(j = 0, \ldots, 4)$ represent five poles of the integrand. Contour for evaluating the generalized integral of this type has been sketched in Fig. 1. In this case, only those poles above the real axis contribute to the integral (16). By applying the theorem of residue we obtain

$$y(x,t) = \text{p.v} \cdot \frac{P}{2\pi EIr_0} \left\{ 2\pi i \sum_{\operatorname{Im}\xi>0} \operatorname{res}\left\{ \frac{\sin r_0\xi \exp[i\xi(x - vt)]}{\prod_{j=0}^{4}(\xi - \xi_j)} \right\} + \pi i \sum_{\operatorname{Im}\xi=0} \operatorname{res}\left\{ \frac{\sin r_0\xi \exp[i\xi(x - vt)]}{\prod_{j=0}^{4}(\xi - \xi_j)} \right\} \right\} \tag{27}$$

in which res$(\cdot)$ represents the residue of the function inside of parenthesis. Since there are poles located on the real axle, integral of this type is evaluated in the sense of Cauchy principle value of integral and symbol "p.v." in Eq. (27) just means that. It should be noted that the first summation term in Eq. (27) requires imaginary part of complex variable ξ positive, which simply says only poles lying in the upper half-plane having contribution to the integration. The second summation term in Eq. (27) applies to those poles that are lying exactly on the real axle.

To expand displacement into closed-form, one needs to express residues in Eq. (27) analytically. To this end, we consider two scenarios. First, for the first summation term where $\mathrm{Im}\,\xi > 0$, we see from aforementioned analysis that two poles are located in the upper complex ξ-plane. In this scenario, the first summation in Eq. (27) is represented as $\sum_{\mathrm{Im}\,\xi>0}(R(\xi)/Q'(\xi))$, where $R(\xi) = \sin r_0\xi \exp[\mathrm{i}\xi(x - vt)]$, $Q(\xi) = \prod_{j=0}^{4}(\xi - \xi_j)$, and $Q'(\xi)$ is the first order derivative of $Q(\xi)$.

For the second summation term, the residue of the pole $\mathrm{Im}\,\xi = 0$ can be obtained by taking the limit to the following formula

$$\mathrm{res}\left[\frac{R(\xi)}{Q(\xi)}\right] = \lim_{\xi\to 0}(\xi - 0)\frac{R(\xi)}{Q(\xi)} \tag{28}$$

Since we have $\lim_{\xi\to 0}\sin(r_0\xi) = 0$, the denominate $R(\xi)$ becomes zero while the denominator remains nonzero valued. This implies that the residue at the pole $\xi = 0$ vanishes. In other words, this pole does not actually contribute to the integration. As a moving point load rather than a moving line load is considered, one just needs to apply the same method to Eq. (17) and take the limit of Eq. (17) with respect to r_0. In the case of $x - vt < 0$, similar method applies except that $\mathrm{Im}\,\xi < 0$ and the contour in the lower half ξ-plane be used.

So far, we have obtained all the solutions corresponding to different cases. If no damping effects are considered, it has been demonstrated in the previous work (Kenney, 1954; Sun, 1998) that waves excited by the moving load propagate along the beam in two directions and it looks symmetric if the observer is traveling with the moving load at the same speed. Waves in two directions correspond respectively to solutions of $x - vt < 0$ and $x - vt > 0$. Mathematically, this requires the poles of the integrand are symmetrically distributed with respect to the real axis. In the elastic cases this is true. However, when hysteretic material damping presents in the system, it is no longer true. In this case, as shown in Fig. 1, these asymmetric poles will result in distinct waves along the positive and negative longitudinal directions. The concrete form of waves can be determined by closed-form solution (27).

6. Numerical computations

Based on the analytical results obtained in previous sections, numerical computations are performed to illustrate dynamic displacement response of beam to a moving load. According to typical pavement structures (Kim and Roesset, 1996), parameters used in calculation are $EI = 2.3$ kN m^2, $K = 68.9$ Mpa, $m = 48.2$ kg/m, $P = 10.5$ kN and $r_0 = 0.075$ m. Fig. 2 plots the displacement variation of beam at position $x = 0$ versus time for different load velocity and damping ratio D. The result shown in Fig. 1 has been compared to the result obtained using finite element method and superposition principle (Kim and Roesset, 1996) and it was found that they are identical. It should also be pointed out that in these figures negative displacement means that the bottom of the beam is suffering with tension stresses.

It is clear from Fig. 2 that damping has visible effect on the shape of the dynamic displacement. The higher the damping ratio, the smaller the absolute value of the maximal displacement. Also, we can see from the displacement that for system with the damping dynamic response is no longer symmetric with respect to time $t = 0$, and the maximal dynamic displacement does not appear exactly beneath the moving load. On the contrary, the maximal displacement occurs at the time after $t = 0$, which implies that the maximal displacement response at $x = 0$ appears behind the moving load. The reason for this phenomenon is because of the damping effect such that the reaction of the beam to the moving load is delayed.

Fig. 3 shows the variation of the maximal displacement with respect to load velocity for beams with different damping. It can be seen that, according to the parameters used here, the critical velocity is $(4KEI/m^2)^{1/4} = 128.5$ m/s at which the maximum of the maximal dynamic displacement is reached. Also, the absolute value of the deflection is growing with increasing velocity of the load at subcritical speed.

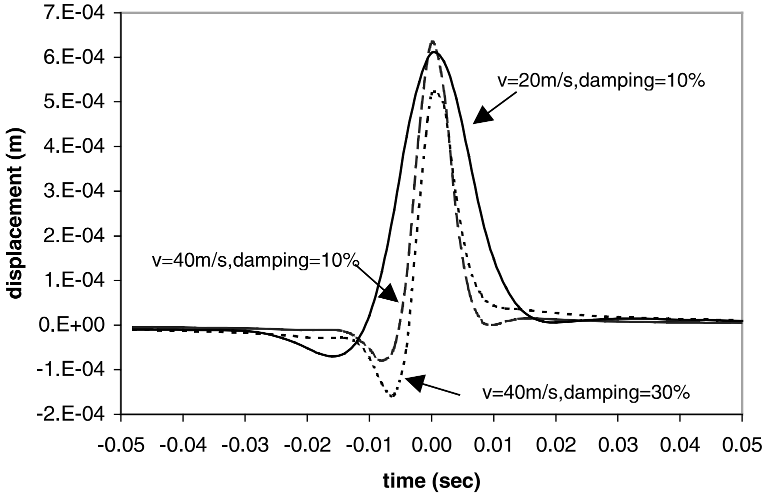

Fig. 2. Dynamic displacement at position $x = 0$ versus time.

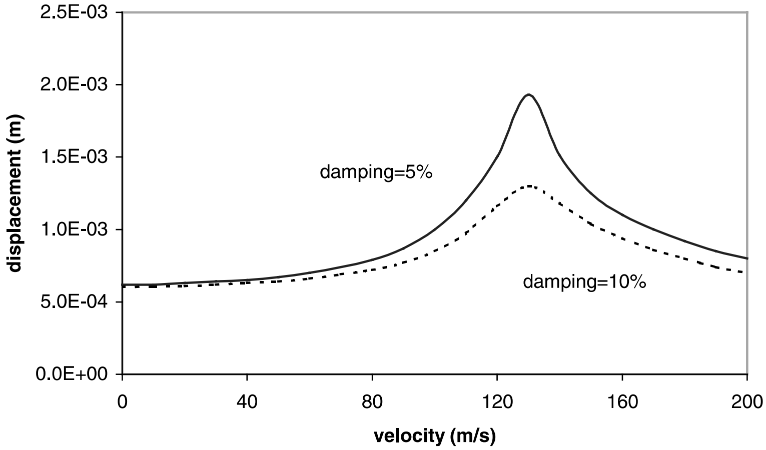

Fig. 3. Maximum displacement of the beam versus velocity.

When the load velocity is less than 128.5 m/s, this conclusion is true. However, at the supercritical speed this conclusion does not hold any more. The maximal dynamic displacement decreases with the increase of load velocity and tends to approaching the maximal static displacement. Although this might have theoretical value in helping us understand the high-speed phenomenon, it might not be of interest in practice because vehicle speed in reality can rarely exceed such a critical velocity barrier.

7. Concluding remarks

In this paper, Fourier transform is used to solve the problem of steady-state response of a beam with linear hysteretic material damping on an elastic foundation subjected to a moving constant line load. The solution is constructed in the form of the convolution of the Green's function of the beam. The theorem of residue is employed to evaluate the generalized integral such that a closed-form solution is achieved. Numerical computations show that displacement response of beam with damping is asymmetric and the maximal displacement of the beam occurs behind the moving load. Also, at the subcritical speed the absolute value of the deflection of the beam increases with the increase of load velocity, while the inverse of this conclusion holds at the supercritical speed. Peak maximal displacement appears when the load travels at the critical speed.

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